

# Dynamics and effective actions of BCS superconductors

 A. van Otterlo<sup>1</sup>, D.S. Golubev<sup>2,4</sup>, A.D. Zaikin<sup>3,4,a</sup>, and G. Blatter<sup>5</sup>
<sup>1</sup> Instituut-Lorentz, Leiden University, PO Box 9506, 2300 RA Leiden, The Netherlands

<sup>2</sup> Physics Department, Chalmers University of Technology, 41296, Göteborg, Sweden

<sup>3</sup> Institut für Theoretische Festkörperphysik, Universität Karlsruhe, 76128 Karlsruhe, Germany

<sup>4</sup> I.E.Tamm Department of Theoretical Physics, P.N. Lebedev Physics Institute, Leninskii pr. 53, 117924 Moscow, Russia

<sup>5</sup> Theoretische Physik, ETH-Hönggerberg, 8093 Zürich, Switzerland

Received 23 November 1998

**Abstract.** We derive the effective dynamical theory for BCS superconductors, based on the effective action formalism. Both the metallic regime  $T \sim T_C$  and the superconducting regime  $T \ll T_C$  are studied in the clean and dirty limit. The full electrodynamics of the problem is formulated in a manifestly gauge-invariant and transparent way. Furthermore, we consider the effect of particle-hole asymmetry in the band structure, and discuss its consequences for vortex dynamics and the topological term in the effective action. The effective action is the starting point for treating (quantum-) dynamical problems involving BCS superconductors.

**PACS.** 74.20.Fg BCS theory and its development – 71.10.-w Theories and models of many electron systems

## 1 Introduction

Dynamical problems in BCS theory [1] are diverse. They include the electromagnetic response of superconductors [2], relaxation phenomena, and collective modes in superconductors [3,4], *e.g.*, the Carlson-Goldman [5] and the Mooij-Schön [6] mode. Further examples are the motion of topological defects, *e.g.*, vortex motion in bulk samples [7–10], quantum tunneling of vortices [11], and thermally activated or quantum phase slips in mesoscopic quasi-one-dimensional wires [12,13], as well as fluctuation effects, *e.g.*, corrections to the conductivity above  $T_C$  [14–16], the renormalization of the critical temperature and of the energy gap of low dimensional dirty superconductors [17,18], and the quantum melting of the vortex lattice [19]. For all these phenomena, an effective (simple) theory of weak coupling BCS superconductivity is desirable. However, such an effective theory is well established only close to and above the critical temperature, where Time Dependent Ginzburg-Landau (TDGL) theory – although only under severe additional restrictions – can be derived from a microscopic starting point [20]. Since the works of Abrahams and Tsuneto [21], Popov [22], Kleiner [23], Ambegaokar [24], and Stoof [25] some is known about the extensions to lower temperatures, but controversies concerning the subject persist. In particular the prefactor of the topological term in the effective action is under discussion, with several papers [26–29] advocating a much larger one than others [9,30,35,41].

Previous studies were mostly restricted to the clean limit at zero temperature and neglected the Coulomb interaction [26–28]. In the present paper, we extend the existing literature in four ways. 1) We fully account for the Coulomb interaction between electrons and between electrons and the ionic Jellium background. 2) We also consider the dirty limit, in which electrons move diffusively rather than ballistically. 3) We allow for particle-hole asymmetry, *i.e.*, the dependence of the density of states on energy, as quantified by its derivative at the Fermi surface  $N'_0 = \partial_\epsilon N(\epsilon_F)$ . The relevance of particle-hole asymmetry for vortex motion and the corresponding Hall-effect has recently been pointed out in references [9,30]. 4) Finally, a guiding principle is called for, as the expansion of the effective action that we will perform (see below) is quite involved. We will make extensive use of gauge invariance and the corresponding Ward-identities. Although perturbation expansions up to a definite order may break gauge-invariance, the Ward identities allow us to construct a manifestly gauge-invariant effective action within perturbation theory.

In references [26–29] Galilei invariance in the clean limit was used as a guiding principle. We argue that in real superconductors Galilei invariance can be broken by at least four mechanisms. 1) Scattering on impurities and phonons break Galilei invariance in a trivial way. 2) Galilei invariance would require a perfect quadratic dispersion, which is not realized in the bandstructure of usual crystal backgrounds. 3) The coupling to electromagnetism, which is Lorentz invariant, also (weakly) breaks Galilei invariance (a  $v_F/c$  effect). 4) At nonzero temperature the gas

---

<sup>a</sup> e-mail: zaikin@tfp.physik.uni-karlsruhe.de

of normal quasi-particle excitations provides a preferred frame of reference which breaks Galilei invariance.

Instead of Galilei invariance, we stress the role of gauge invariance. The corresponding Ward identities express particle number and charge conservation and can be used to rewrite the effective action in a manifestly gauge-invariant way. This means that in the superconducting phase the effective action depends only on the amplitude of the energy gap  $|\Delta|$ , the superfluid velocity  $\mathbf{v}_s = \frac{1}{2m}(\nabla\varphi - \frac{2e}{c}\mathbf{A})$ , the chemical potential for Cooper pairs  $\tilde{\Phi} = V - \dot{\varphi}/2e$ , and the electromagnetic fields  $\mathbf{E}$  and  $\mathbf{B}$ . This is in accord with the Anderson-Higgs mechanism [31]: the superconducting phase  $\varphi$  does not appear explicitly in the effective action. We will see below that, for example, if the equilibrium value of the energy gap  $|\Delta_0|$  does not depend on space and time coordinates and all fields vary slowly in space and time, the action in the particle-hole symmetric case has the form

$$S = \int \frac{d\mathbf{q}}{2} \left( \frac{\epsilon \mathbf{E}^2 + \mathbf{B}^2/\mu}{4\pi} + \chi_A |\Delta_1|^2 + \chi_L \mathbf{v}_s^2 + \chi_J \tilde{\Phi}^2 \right). \quad (1)$$

Here  $|\Delta_1|$  is the small deviation of the order parameter from the equilibrium value, the  $\chi$ 's are generalized susceptibilities,  $\epsilon$  is the dielectric function of the metal, and  $\mu$  is the magnetic permeability. The terms involving  $\mathbf{E}$  and  $\mathbf{B}$  include the electronic polarization contributions to the electromagnetic fields that will be discussed below in Section 3. The term involving the amplitude fluctuations  $|\Delta_1|$  is discussed in Section 4. Most interesting are the last two terms: the first describes how the gradient of the superconducting phase  $\nabla\varphi$  tries to adjust to the local vector potential  $\mathbf{A}$ . The prefactor  $\chi_L$  is proportional to the superfluid density  $n_s$  and this term is just the kinetic energy of the superfluid, related to the DC Josephson effect and to London theory. Similarly, the less familiar second term describes how the time derivative of the superconducting phase  $\dot{\varphi}$  tries to adjust to the local scalar potential  $V$ . This is the term that produces the AC Josephson effect. The prefactor  $\chi_J$  is proportional to the superfluid density as well. If the critical temperature  $T_C$  is approached from below, the superfluid density vanishes, and the chemical potential for Cooper pairs  $\tilde{\Phi}$  and the chemical potential for quasi-particles  $V$  decouple. This last term also describes how the motion of vortices in the mixed state leads to a voltage drop across the sample *via* the time dependence of the phase.

In case particle-hole symmetry is broken, an additional contribution to the action arises, which is proportional to the derivative of the density of states at the Fermi energy  $N'_0 = \partial_\epsilon N(\epsilon_F)$ . The dimensionless parameter that characterizes the amount of particle-hole symmetry breaking is  $\gamma = \Delta N'_0 / (2\lambda N_0^2)$ , with BCS coupling constant  $\lambda$ . In usual weak coupling superconductors with  $\lambda N_0 \sim 10^{-1}$  and  $\Delta/\epsilon_F \sim 10^{-3}$  the parameter  $\gamma$  is rather small,  $\gamma \sim 10^{-2}$ . Nevertheless, it is important for vortex motion, and its possible relation to the sign-anomaly in the Hall effect has been discussed in references [9, 30]. The general form

of the particle-hole asymmetric part in the action is

$$S = -2ieN_0\Gamma \int dx \tilde{\Phi} (|\Delta_0 + \Delta_1|^2 - |\Delta_0|^2), \quad (2)$$

with  $\Gamma = \gamma/\Delta_0$ . The physical origin of this term is the coupling of the electronic density to the energy gap when particle-hole symmetry is broken. Thus, fluctuations in the amplitude of the gap cause charge density fluctuations, which couple directly to the potential  $\tilde{\Phi}$ , see also Section 4 for a discussion of this point.

Our discussion below will be within the imaginary time Matsubara formalism. Here we will not address the important point of relaxation mechanisms like spin-flip, electron-electron, and electron-phonon scattering that could be accounted for. We will simply assume that some relaxation mechanism is available which brings our superconducting system into equilibrium with a big reservoir. We would like to emphasize that this assumption is by no means in contradiction with the main goal of our paper: to provide a convenient approach for studying dynamical and nonequilibrium phenomena in superconductors. Rather, it restricts the scope of the phenomena which can be effectively described with our methods.

In what physical situations is our imaginary time formalism meant to work? One such situation is quite standard: a superconductor only slightly driven out of equilibrium, so that one can describe nonequilibrium effects within a linear response theory and express the results in terms of equilibrium correlation functions, which – after a proper analytic continuation – will describe the dynamics of the system in real time. Another important class of phenomena is related to (quantum) fluctuations of the superconducting order parameter, both of its modulus and its phase. Such fluctuations may – and in general do – involve *virtual* states with the electron subsystem driven *far from equilibrium*. Examples are quantum phase slips in thin superconducting wires [13] and quantum tunneling of vortices [11]. All such processes are also conveniently described within the formalism developed below.

The analysis of the real time dynamics of a superconductor with strong deviations of the quasiparticle distribution function from equilibrium in general requires methods based on the Keldysh technique that keep track explicitly of the distribution function, see reference [32]. Strong nonequilibrium real time dynamics is beyond the scope of the present paper and will be discussed elsewhere. Here we would only like to point out that many features of our imaginary time analysis can be directly generalized and used also within the real time Keldysh technique. Thus, also for the real time case a lot can be learned already from the present imaginary time formulation of the problem.

The calculation that leads to the effective actions (1, 2) will be presented in the next section. Its physical content is discussed in Sections 3 (normal state) and 4 (superconducting state). Where possible, we give the explicit forms of the propagators of the various fields in the hydrodynamic limit (in Sects. 3 and 4), whereas the more general (and more complicated) expressions are deferred to Appendix B, where also the calculation of polarization

bubbles is outlined. The derivation of the Ward identities is given in Appendix A.

Parts of the present paper were implicit in references [9,13] and to a somewhat lesser extent in references [17,22–24,33,34].

## 2 Model and derivation of effective action

The starting point for our analysis is a model Hamiltonian that includes a short range attractive weak coupling BCS and a long range repulsive Coulomb interaction. We represent the latter in terms of the fluctuating gauge fields of electro-magnetism,  $V$  and  $\mathbf{A}$ . The idea is to integrate out the electronic degrees of freedom on the level of the partition function, leaving us with an effective theory in terms of collective fields [22–24,33,34]. The partition function  $Z$  is conveniently expressed as a path integral over the anticommuting electronic fields  $\bar{\psi}$ ,  $\psi$  and the commuting gauge fields  $V$  and  $\mathbf{A}$ , together with a gauge condition. The Euclidean action reads

$$S = \int dx \left( \bar{\psi}_\sigma [\partial_\tau - ieV + \xi(\nabla - \frac{ie}{c}\mathbf{A})] \psi_\sigma - \lambda \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow + ien_i V + [\mathbf{E}^2 + \mathbf{B}^2]/8\pi \right). \quad (3)$$

Here  $\xi(\nabla) \equiv -\nabla^2/2m - \mu$  describes a single conduction band with quadratic dispersion,  $\lambda$  is the BCS coupling constant,  $\sigma = \uparrow, \downarrow$  is the spin index, and  $en_i$  denotes the background charge density of the ions. In our notation  $dx$  denotes  $d^3\mathbf{x}d\tau$  and we use units in which  $\hbar$  and  $k_B$  are set equal to unity. The field strengths are functions of the gauge fields through  $\mathbf{E} = -\nabla V + (1/c)\partial_\tau \mathbf{A}$  and  $\mathbf{B} = \nabla \times \mathbf{A}$  in the usual way for the imaginary time formulation.

We use a Hubbard-Stratonovich transformation to decouple the BCS interaction term and to introduce the superconducting energy gap  $\Delta = |\Delta|e^{i\varphi}$  as an order parameter

$$\exp \left( \lambda \int dx \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right) = \left[ \int \mathcal{D}^2 \Delta e^{-\lambda^{-1} \int dx |\Delta|^2} \right]^{-1} \times \int \mathcal{D}^2 \Delta e^{-\int dx (\lambda^{-1} |\Delta|^2 + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow + \bar{\Delta} \psi_\downarrow \psi_\uparrow)}, \quad (4)$$

where the first factor is for normalization and will not be important in the following. As a result, the partition function now reads

$$Z = \int' \mathcal{D}^2 \Delta D V \mathcal{D}^3 \mathbf{A} \mathcal{D}^2 \Psi \exp \left( -S_0 - \int dx \bar{\Psi} \mathcal{G}^{-1} \Psi \right),$$

$$S_0[V, \mathbf{A}, \Delta] = \int dx \left( \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} + ien_i V + \frac{|\Delta|^2}{\lambda} \right), \quad (5)$$

where the prime on the integral denotes the restriction to a certain gauge choice for the electromagnetic potentials  $V$  and  $\mathbf{A}$ . Below in Section 3, we will sometimes use the Coulomb gauge  $\nabla \cdot \mathbf{A} \equiv 0$  in which the vector potential is

completely transverse. In equation (5) we have also introduced the Nambu spinor notation for the electronic fields

$$\Psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \quad \bar{\Psi} = (\bar{\psi}_\uparrow \ \bar{\psi}_\downarrow) \quad (6)$$

and the matrix Green's function in Nambu space

$$\mathcal{G}^{-1} = \begin{pmatrix} \partial_\tau - ieV + \xi(\nabla - \frac{ie}{c}\mathbf{A}) & \Delta \\ \bar{\Delta} & \partial_\tau + ieV - \xi(\nabla + \frac{ie}{c}\mathbf{A}) \end{pmatrix};$$

$$\mathcal{G} = \begin{pmatrix} G & F \\ \bar{F} & \bar{G} \end{pmatrix}, \quad (7)$$

with normal and anomalous Green's functions denoted by  $G$  and  $F$ .

After a final Gaussian integration over the electronic degrees of freedom, we are left with the effective action

$$S_{\text{eff}} = -\text{Tr} \ln \mathcal{G}^{-1} + S_0[V, \mathbf{A}, \Delta]. \quad (8)$$

Here, the trace “Tr” denotes both a matrix trace in Nambu space and a trace over internal coordinates or momenta and frequencies. In the following “tr” is used to denote a trace over internal coordinates only.

### 2.1 The equations of motion

The Euler-Lagrange equations obtained by varying the action (8) with respect to  $V$  and  $\mathbf{A}$  yield the two Maxwell equations that describe Thomas-Fermi and London screening, respectively. They read

$$\nabla \cdot \mathbf{E} = 4\pi e [n_e - n_i],$$

$$-\frac{1}{c} \partial_\tau \mathbf{E} + \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}_e. \quad (9)$$

Note that the ionic background contributes only to the charge density, and not to the current, if we describe the system in the frame where the ions are at rest. Both the electronic density  $n_e$  and current density  $\mathbf{j}_e$  are expressed through the diagonal elements  $G$  and  $\bar{G}$  of the matrix (in Nambu space) electron Green's functions  $\mathcal{G}$ . Explicitly,

$$n_e(x) = \text{Tr}[\mathcal{G}\sigma_3] = \bar{G}(x, x) - G(x, x),$$

$$\mathbf{j}_e(x) = \frac{e}{m} \text{Tr}[(i\nabla \mathbf{1} + \frac{e}{c} \mathbf{A} \sigma_3) \mathcal{G}] \quad (10)$$

$$= \frac{e}{m} \left[ (i\nabla + \frac{e}{c} \mathbf{A}) G(x, y) + (i\nabla - \frac{e}{c} \mathbf{A}) \bar{G}(x, y) \right]_{y=x}.$$

The matrices  $\sigma_{1,2,3}$  are the Pauli matrices and below we will also use  $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ .

The electronic density  $n_e$  is a function of the chemical potential  $\mu$ , and in the presence of particle-hole asymmetry also of the energy gap  $\Delta$ . At zero temperature it satisfies  $n_e(\mu = \epsilon_F, \Delta = 0) = n_i$ . In general the electronic density can be expanded as  $n_e(\mu + ieV, \Delta) = n_i + 2ieN_0V + 2N_0\Gamma\Delta^2 + \dots$ , with the density of states per spin  $N_0$  and the particle-hole asymmetry parameter

$\Gamma = \partial_\epsilon N(\epsilon_F)/(2\lambda N_0^2)$ , see reference [35]. The requirement of overall charge neutrality makes the electrostatic potential  $V$  a function of the energy gap,  $V_\Delta = -i\Gamma\Delta^2/e$ . Longitudinal electric fields and deviations of the electronic density  $n_e$  from the ionic density  $n_i$  are screened on the Thomas-Fermi length scale  $\lambda_{\text{TF}}^{-2} = 8\pi e^2 N_0$ . In the superconducting state, in addition the magnetic field  $\mathbf{B}$  is screened on the scale of the London penetration depth  $\lambda_L^{-2} = 4\pi e^2 n_s/mc^2$ , where  $n_s$  denotes the superfluid density.

Varying the action (8) with respect to  $\bar{\Delta}$  yields the BCS gap-equation for  $\Delta$

$$\Delta(x) = \text{Tr}[\mathcal{G}\sigma_-] = \lambda F(x, x), \quad (11)$$

with the anomalous Green's function  $F$ . The gap equation has a constant solution  $\Delta_0 = |\Delta_0| \exp(i\varphi_0)$  as well as more complex time and space dependent solutions, such as vortices.

## 2.2 Perturbation expansion

The effective action in equation (8) is the starting point for an expansion around the constant saddle point solution  $\Delta = \Delta_0$ ,  $V = V_\Delta$ , and  $\mathbf{A} = 0$ . We absorb the constant  $V_\Delta$  in the chemical potential  $\mu$  from now on, so that it doesn't appear explicitly in the following.

There are two ways of organizing the perturbation expansion. In this section we will expand in  $V$ ,  $\mathbf{A}$ , and  $\Delta_1 = \Delta - \Delta_0$ , and to this end split the inverse Green's function in equation (7) into an unperturbed part  $\mathcal{G}_0^{-1}$  and a perturbation  $\mathcal{G}_1^{-1}$ , according to

$$\mathcal{G}_0 = \begin{pmatrix} G & F \\ \bar{F} & \bar{G} \end{pmatrix}, \quad \mathcal{G}_1^{-1} = \begin{pmatrix} K - L & \Delta_1 \\ \bar{\Delta}_1 & -K - L \end{pmatrix},$$

$$K = \frac{m}{2} \left( \frac{e}{mc} \right)^2 \mathbf{A}^2 - ieV, \quad L = \frac{-i}{2} \frac{e}{mc} \{ \nabla, \mathbf{A} \}, \quad (12)$$

where  $\{.,.\}$  denotes an anti-commutator. In the following it is understood that the unperturbed Green's function has a chemical potential  $\mu + ieV_\Delta$  and an energy gap  $\Delta_0$ . Without loss of generality, we choose  $\Delta_0$  to be real.

At first sight the splitting we have just made seems not convenient and to restrict severely the generality of our analysis, since we expand around a state with constant phase of the order parameter. However, this does not mean that we exclude, *e.g.*, current carrying states for which the phase of  $\Delta$  depends on coordinates in an essential way and is not small everywhere in the superconductor. We show below that the actual parameters of our expansion are the gauge invariant linear combinations of the electromagnetic potentials and the phase of the order parameter,  $\Phi = V - \dot{\varphi}/2e$  and  $\mathbf{v}_s = \frac{1}{2m}(\nabla\varphi - \frac{2e}{c}\mathbf{A})$ . Only these parameters (and not  $V$ ,  $\mathbf{A}$ , and  $\varphi$  separately) are required to be small within the framework of our analysis. Thus, also states that carry a current which is not necessarily small can be described. This is a direct consequence of gauge invariance which plays an important role in our consideration.

The other way of organizing the expansion is commented upon in Subsection 2.5. It involves a unitary gauge transformation of the fields, after which one expands directly in the gauge invariant fields  $\Phi$  and  $\mathbf{v}_s$ , and a manifestly real perturbation  $\Delta_1 = |\Delta_1|$ . With the help of a Ward-identity, the two expansions can be shown to be fully equivalent. For pedagogical purposes we postpone the corresponding discussion (see Subject. 5) and proceed with the expansion.

The trace of the inverse Green's function can be expanded in  $\mathcal{G}_1^{-1}$  using

$$\text{Tr} \ln \mathcal{G}^{-1} = \text{Tr} \ln \mathcal{G}_0^{-1} + \text{Tr} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\mathcal{G}_0 \mathcal{G}_1^{-1})^n, \quad (13)$$

and only terms of order  $n = 1$  and  $2$  will be needed here.

The  $n = 1$  term in the effective action is  $-\text{Tr}(\mathcal{G}_0 \mathcal{G}_1^{-1})$ . The explicit evaluation of the trace yields

$$S_1 = -\text{tr}[K(G - \bar{G}) - L(G + \bar{G}) + \bar{\Delta}_1 F + \Delta_1 \bar{F}] \quad (14)$$

$$= \int dx \left( -ien_e V + \frac{ne^2}{2mc^2} \mathbf{A}^2 - \frac{\bar{\Delta}_1 \Delta_0 + \Delta_1 \bar{\Delta}_0}{\lambda} \right),$$

where we have used  $G(x, x) = -\bar{G}(x, x) = -n_e/2$ , and  $F(x, x) = \Delta_0/\lambda$  according to the gap equation.

The second order contribution  $S_2 = \frac{1}{2} \text{Tr}(\mathcal{G}_0 \mathcal{G}_1^{-1})^2$  reads

$$S_2 = \frac{1}{2} \text{tr}[GKGK + \bar{G}K\bar{G}K - 2FK\bar{F}K + 2FL\bar{F}L$$

$$+ GLGL + \bar{G}L\bar{G}L - 2GKGL + 2\bar{G}K\bar{G}L - 2\bar{F}KFL$$

$$+ 2FK\bar{F}L + 2F\bar{\Delta}_1 GK - 2\bar{F}\Delta_1 \bar{G}K + 2G\Delta_1 \bar{F}K$$

$$- 2\bar{G}\bar{\Delta}_1 FK - 2F\bar{\Delta}_1 GL - 2\bar{F}\Delta_1 \bar{G}L - 2G\Delta_1 \bar{F}L$$

$$- 2\bar{G}\bar{\Delta}_1 FL + \bar{F}\Delta_1 \bar{F}\Delta_1 + F\bar{\Delta}_1 F\bar{\Delta}_1 + 2G\Delta_1 \bar{G}\bar{\Delta}_1]. \quad (15)$$

As we are interested in contributions up to second order in the fields, it suffices to take  $K = -ieV$  in this expression.

## 2.3 Longitudinal and transverse physics

The next step is the evaluation of the traces in the expansion equation (15). For two Green's functions  $G$  and  $G'$ , and two fields  $A$  and  $A'$ , the following identities hold:

$$\text{tr}[GAG'A'] = \int \bar{d}q A(q) A'(-q) \{1\}_{GG'};$$

$$\text{tr}[GAG'\{\nabla_a, A'_a\}] =$$

$$2i \int \bar{d}q A(q) A'_a(-q) \{(p+q/2)_a\}_{GG'}; \quad (16)$$

$$\text{tr}[G\{\nabla_a, A_a\}G'\{\nabla_b, A'_b\}] =$$

$$-4 \int \bar{d}q A_a(q) A'_b(-q) \{(p+q/2)_a(p+q/2)_b\}_{GG'},$$

where  $a = x, y, z$  (repeated indices are summed over) and we have introduced the short hand notation  $q = (\mathbf{q}, \omega_\mu)$ ,

$\int \mathrm{d}q \equiv T \sum_{\omega, \mu} \int \mathrm{d}^3 \mathbf{q} / (2\pi)^3$ , as well as the bracket notation for polarization bubbles

$$\{B\}_{GG'}(q) = \int \mathrm{d}p B G(p+q) G'(p). \quad (17)$$

Furthermore, we will split all fields in longitudinal and transverse components with respect to the momentum  $\mathbf{q}$  by making use of the corresponding projection operators

$$P_L^{ab} = q^a q^b / \mathbf{q}^2, \quad P_T^{ab} = \delta^{ab} - q^a q^b / \mathbf{q}^2 \quad (18)$$

that satisfy  $P_i^2 = P_i$ ,  $P_L P_T = P_T P_L = 0$ , and  $P_L + P_T = 1$ . With the help of the projection operators any vector field  $V^a(\mathbf{q})$  can be decomposed into a longitudinal part  $V_L^a(\mathbf{q}) = P_L^{ab} V^b(\mathbf{q}) = (\mathbf{V} \cdot \mathbf{q} / \mathbf{q}^2) q^a$  and a transverse part  $V_T^a(\mathbf{q}) = P_T^{ab} V^b(\mathbf{q}) = V^a - (\mathbf{V} \cdot \mathbf{q} / \mathbf{q}^2) q^a$ . Similarly, tensors  $T^{ab}(\mathbf{q})$  are decomposed into  $T_L^{ab}(\mathbf{q}) = \mathrm{Tr}[P_L T] P_L^{ab}$  and  $T_T^{ab}(\mathbf{q}) = \frac{1}{2} \mathrm{Tr}[P_T T] P_T^{ab}$ , where the extra half in the latter arises since  $\mathrm{Tr} P_T = d - 1 = 2$ , in contrast to  $\mathrm{Tr} P_L = 1$ .

We use these considerations to simplify the polarization bubbles that we encounter in the perturbation expansion. By splitting  $(p+q/2)_a$  into longitudinal and transverse components, we decompose the bubbles as

$$\begin{aligned} \{(p+q/2)_a\}_{GG'} &= \{Q/\mathbf{q}^2\}_{GG'} q^a, \\ \{(p+q/2)_a(p+q/2)_b\}_{GG'} &= \{Q^2/\mathbf{q}^2\}_{GG'} P_L^{ab} \\ &\quad + \{(\mathbf{p} \times \mathbf{q})^2 / 2\mathbf{q}^2\}_{GG'} P_T^{ab}, \end{aligned} \quad (19)$$

where in order to simplify notation, we have introduced  $Q = \mathbf{q} \cdot (\mathbf{p} + \mathbf{q}/2)$ . In the following, we need only specific polarization bubbles for which we introduce the notation

$$\begin{aligned} g_0 &= \{1\}_{GG}(q), \quad g_1 = \{Q/\mathbf{q}^2\}_{GG}(q), \quad (20) \\ g_2 &= \{Q^2/\mathbf{q}^4\}_{GG}(q), \quad g_3 = \{(\mathbf{p} \times \mathbf{q})^2 / 2\mathbf{q}^4\}_{GG}(q). \end{aligned}$$

Analogously the  $f_i, h_i, k_i$  denote  $\{..\}_{FF}$ ,  $\{..\}_{G\bar{G}}$ , and  $\{..\}_{FG}$  respectively.

Finally, as pointed out in reference [17], it is advantageous to also split the fluctuations of the energy gap in real longitudinal and transverse components  $\Delta_1 = \Delta_L + i\Delta_T$ . We will see in the next subsection that  $\Delta_L$  and  $\Delta_T$  appear in the effective action in rather different ways, related to their different physical nature.

## 2.4 The effective action

The effective action up to second order in the fields  $\Delta_L$ ,  $\Delta_T$ ,  $V$ , and  $\mathbf{A}$  is found by gathering the terms from  $S_0$ ,  $S_1$ , and  $S_2$  (Eqs. (5, 14, 15)). We split  $S_{\mathrm{eff}}$  into a constant mean field part  $S_{\mathrm{eff}}^0$  and a Gaussian fluctuation part  $S_{\mathrm{eff}}^2$ . No first order contribution is present, as the terms  $i\pi n_i V$  and  $-i\pi n_e V$  cancel by charge neutrality. We find

$$\begin{aligned} S_{\mathrm{eff}}^0 &= -\mathrm{Tr} \ln \mathcal{G}_0^{-1}[\Delta_0] + \beta \mathcal{V} \Delta_0^2 / \lambda, \quad (21) \\ S_{\mathrm{eff}}^2 &= \int \mathrm{d}q \left[ \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} + \frac{\Delta_L^2 + \Delta_T^2}{\lambda} + \frac{nm}{2} \left( \frac{e}{mc} \right)^2 \mathbf{A}^2 \right. \\ &\quad \left. + \left( \Delta_L \Delta_T eV \frac{eA^a}{mc} \right)_q \hat{\mathcal{M}}_q \begin{pmatrix} \Delta_L \\ \Delta_T \\ eV \\ eA^b/mc \end{pmatrix}_{-q} \right], \end{aligned}$$

where  $\mathcal{V}$  denotes the volume of the system,  $\beta$  the inverse temperature, and we have introduced a matrix notation with

$$\begin{aligned} \hat{\mathcal{M}}_q &= \begin{pmatrix} h_0 + f_0 & -ih_0 & -2ik_0 & -2q^a k_1 \\ ih_0 & h_0 - f_0 & -2k_0 & 2iq^b k_1 \\ -2ik_0 & 2k_0 & f_0 - g_0 & iq^b g_1 \\ -2q^a k_1 & -2iq^a k_1 & iq^a g_1 & m^{ab} \end{pmatrix}, \\ m^{ab} &= \mathbf{q}^2 [(g_3 + f_3) P_T^{ab} + (g_2 + f_2) P_L^{ab}]. \end{aligned} \quad (22)$$

In the above expression for the matrix  $\hat{\mathcal{M}}_q$  it is understood that all kernels are taken at momentum and frequency  $q$ .

The physical content of the effective action (21) can be brought out more clearly by ‘‘diagonalizing the matrix’’, *i.e.*, rewriting equation (21) in terms of the eigenmodes. To this end we introduce the superfluid velocity and the chemical potential for Cooper pairs in terms of combinations of the gauge fields and the transverse gap fluctuations as  $\mathbf{v}_s = \frac{1}{2m} [(\nabla \Delta_T) / \Delta_0 - \frac{2e}{c} \mathbf{A}]$  and  $\Phi = V - \frac{1}{2e} \dot{\Delta}_T / \Delta_0$ .

The terms in equation (21) that couple the transverse phase-like gap fluctuations to the other fields assume a diagonal form with the use of the Ward-identity (A.8). Little algebra shows explicitly that

$$\begin{aligned} &[\lambda^{-1} + h_0 - f_0] \Delta_T^2 + 4k_0 V(q) \Delta_T(-q) \\ &- (4ie/mc) k_1 \mathbf{q} \cdot \mathbf{A}(q) \Delta_T(-q) \equiv (4e^2 \Delta_0 k_0 / i\omega_\mu) [\Phi^2 - V^2] \\ &\quad + 4m \Delta_0 k_1 P_L^{ab} [(e^2 A^a A^b / m^2 c^2) - v_s^a v_s^b], \end{aligned}$$

which we use to eliminate  $\Delta_T$  from the action. Furthermore, the field strengths are invariant under gauge transformations, so that they may be expressed in  $\Phi$  and  $\mathbf{v}_s$  as

$$\begin{aligned} |\mathbf{E}|^2 &= \mathbf{q}^2 |\Phi(q)|^2 + \frac{m^2 \omega^2}{e^2} |\mathbf{v}_s(q)|^2 \\ &\quad + \frac{m\omega}{e} [\Phi(q) \mathbf{q} \cdot \mathbf{v}_s(-q) + \Phi(-q) \mathbf{q} \cdot \mathbf{v}_s(q)] \\ |\mathbf{B}|^2 &= \frac{m^2 c^2}{e^2} \mathbf{q}^2 P_T^{ab} v_s^a(q) v_s^b(-q), \end{aligned} \quad (23)$$

which allows us to rewrite the terms related to  $\mathbf{A}$  and  $V$  in terms of  $\mathbf{E}$  and  $\mathbf{B}$ . Finally, due to the Ward-identities (A.6, A.7), the remaining terms in  $\mathbf{A}$  and  $V$  are seen to vanish. Together with the propagator  $\chi_A(q) = [2\lambda^{-1} + h_0(q) + h_0(-q) + 2f_0(q)]$  for the longitudinal gap fluctuations and introducing the London and Josephson susceptibilities  $\chi_L(q) = -8m \Delta_0 k_1(q)$  and  $\chi_J(q) = 8e^2 \Delta_0 k_0(q) / i\omega_\mu$ , we obtain the ‘‘normal’’ and the ‘‘superconducting’’ contributions to the effective action

$$\begin{aligned} S_{\mathrm{sc}}[\Delta_L, \Phi, \mathbf{v}_s] &= -\mathrm{Tr} \ln (\mathcal{G}_0^{-1}[\Delta_0] \mathcal{G}_0[0]) + \beta \mathcal{V} \Delta_0^2 / \lambda \\ &\quad + \frac{1}{2} \int \mathrm{d}q (\chi_A \Delta_L^2 + \chi_J \Phi^2 + \chi_L \mathbf{v}_s^2), \end{aligned} \quad (24)$$

$$\begin{aligned} S_{\mathrm{nm}}[\mathbf{E}, \mathbf{B}] &= -\mathrm{Tr} \ln \mathcal{G}_0^{-1}[0] \\ &\quad + \frac{1}{2} \int \mathrm{d}q (\chi_E \mathbf{E}^2 - \chi_M \mathbf{B}^2). \end{aligned} \quad (25)$$

Here we have also introduced the electric and magnetic susceptibilities  $\chi_E(q) = 2e^2 g_1(q)/(mi\omega_\mu)$  and  $\chi_M(q) = 2e^2[g_2(q) + f_2(q) - g_3(q) - f_3(q)]/(m^2 c^2)$ .

The terms in equation (21) that couple the longitudinal gap fluctuations to the other fields are nonvanishing only if particle-hole symmetry is broken. Again by virtue of the Ward-identity equation (A.8), they combine into the action  $S_{\text{ph}}$  that describes the effects of particle-hole symmetry breaking

$$S_{\text{ph}} = \int \tilde{d}q \left( \frac{1}{2} \chi_R(q) \Delta_L(q) \Delta_L(-q) + \chi_R^\parallel(q) \Phi(q) \Delta_L(-q) + \chi_R^\perp(q) \mathbf{q} \cdot \mathbf{v}_s(q) \Delta_L(-q) \right), \quad (26)$$

where we have introduced the susceptibilities  $\chi_R(q) = h_0(q) - h_0(-q)$ ,  $\chi_R^\parallel(q) = -2ie[k_0(q) + k_0(-q)]$ , and  $\chi_R^\perp(q) = 2[k_1(q) - k_1(-q)]$ .

Defining

$$S_{\text{em}}[\mathbf{E}, \mathbf{B}] = \int \tilde{d}q \left( \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} \right), \quad (27)$$

we obtain the final result

$$S_{\text{eff}} = S_{\text{sc}}[\Delta_L, \Phi, \mathbf{v}_s] + S_{\text{nm}}[\mathbf{E}, \mathbf{B}] + S_{\text{ph}}[\Delta_L, \Phi, \mathbf{v}_s] + S_{\text{em}}[\mathbf{E}, \mathbf{B}]. \quad (28)$$

Using the standard definitions  $\epsilon = 1 + 4\pi\chi_E$  and  $\mu^{-1} = 1 - 4\pi\chi_M$ , we arrive at the form (1) quoted in the introduction. The expressions (24–28) represent the main result of this section, and a convenient starting point for any study of dynamical processes.

Note that in arriving at equation (28) the terms  $\pm ien_i V$  from  $S_0$  and  $S_1$  have cancelled, since on the average the electronic and ionic charge densities cancel. This point was not appreciated in references [27–29, 36], where no coupling to electromagnetism was included and only the term from  $S_1$  was found.

We have decomposed the action into four parts: the superconducting contribution  $S_{\text{sc}}$ , the normal metallic contribution  $S_{\text{nm}}$ , the particle-hole symmetry breaking action  $S_{\text{ph}}$ , and the action of the free electromagnetic fields  $S_{\text{em}}$ . Let us emphasize that the possibility of such a decomposition is a direct consequence of the Ward identities. As these identities follow from gauge invariance only and do not depend on the presence and concentration of impurities in a superconductor, we conclude that the splitting of the full action into four parts in equation (28) holds not only for clean superconductors but rather for an arbitrary concentration of impurities.

## 2.5 Gauge invariance

At this stage it is appropriate to discuss the consequences of gauge invariance for the effective action (28) a bit more deeply, see also Appendix A. Inspecting equation (28), we observe that the transverse component of the energy gap  $\Delta_T$  has completely disappeared from the effective action.

This is just the Anderson-Higgs mechanism [31]: the Goldstone mode  $\Delta_T$  is “gauged away” and appears only within the combinations  $\Phi$  and  $\mathbf{v}_s$ . Since the electromagnetic field strengths  $\mathbf{E}$  and  $\mathbf{B}$  have been expressed in terms of  $\Phi$  and  $\mathbf{v}_s$ , the integral over the field  $\Delta_T$  in the partition function factorizes and contributes an irrelevant constant.

The partition function  $Z$  can be represented in two equivalent ways. In the first, the Goldstone mode is explicitly present and the four gauge field components are restricted by a gauge condition. In the second, the Goldstone mode is “eaten” by the gauge condition and the four gauge field components are unrestricted. Explicitly

$$\int' \mathcal{D}\Delta_L \mathcal{D}\Delta_T \mathcal{D}V \mathcal{D}^3 \mathbf{A} \exp(-S_{\text{eff}}[\Delta_L, \Delta_T, V, \mathbf{A}]) \equiv \int \mathcal{D}\Delta_L \mathcal{D}\Phi \mathcal{D}^3 \mathbf{v}_s \exp(-S_{\text{eff}}[\Delta_L, \Phi, \mathbf{v}_s]), \quad (29)$$

where the prime on the first integral denotes that it is supplemented by a gauge condition. In both cases 5 dynamical degrees of freedom are present.

We now return to the point raised in Subsection 2.2 concerning the two possible ways of organizing the expansion. Observe that the trace of the inverse Green’s function, which is the starting point of the perturbation expansion, is invariant under unitary transformations

$$\text{Tr} \ln \mathcal{G}^{-1} = \text{Tr} \ln \mathcal{U} \mathcal{G}^{-1} \mathcal{U}^{-1}. \quad (30)$$

Choosing  $\mathcal{U}(\theta) = \exp(-i\sigma_3\theta/2)$  with an arbitrary space and time dependent function  $\theta(x)$ , we obtain from the old Green’s function (7), the new inverse Green’s function  $\tilde{\mathcal{G}}^{-1} = \exp(-i\sigma_3\theta/2)\mathcal{G}^{-1}\exp(i\sigma_3\theta/2)$  that reads

$$\tilde{\mathcal{G}}^{-1} = \begin{pmatrix} \partial_\tau - ie\Phi + \xi(\nabla + im\mathbf{v}_s) & e^{-i\theta} \Delta \\ e^{i\theta} \Delta & \partial_\tau + ie\Phi - \xi(\nabla - im\mathbf{v}_s) \end{pmatrix}. \quad (31)$$

If  $\Delta = |\Delta|e^{i\varphi}$ , such a gauge transformation can be used to make the energy gap in the Green’s function real by choosing  $\theta = \varphi$ . Instead, the superconducting phase appears in the chemical potential for Cooper pairs  $\Phi = V - \dot{\varphi}/2e$  that replaces  $V$  and in the superfluid velocity  $\mathbf{v}_s = \frac{1}{2m}(\nabla\varphi - \frac{2e}{c}\mathbf{A})$  that replaces  $-e\mathbf{A}/mc$ . Thus, we may identify  $\varphi$  with  $\Delta_T/\Delta_0$  from the previous subsection.

Note that the unitary operator  $\mathcal{U}(\theta)$  is related to a local  $U(1)$  gauge transformation

$$\begin{aligned} V &\rightarrow V - \frac{1}{2e}\dot{\theta}; \\ \mathbf{A} &\rightarrow \mathbf{A} - \frac{c}{2e}\nabla\theta; \\ \Psi &= \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \rightarrow \mathcal{U}(\theta)\Psi = \begin{pmatrix} e^{-i\theta/2}\psi_\uparrow \\ e^{i\theta/2}\psi_\downarrow \end{pmatrix}; \\ \Delta &\rightarrow e^{-i\theta}\Delta, \end{aligned} \quad (32)$$

that leaves the field strengths invariant. Since the phase of *all* charged fields is rotated by a gauge transformation, one also should replace the term  $ien_i V$  in equation (5) by

$ien_i\Phi$ . The point being, that the ionic background charge density is eventually made up by particles and corresponding fields as well. In principle, one should describe the positively charged ions in the starting point (3) by a field  $\psi_i$ , and include a term  $\bar{\psi}_i[\partial_\tau + ieV]\psi_i$  in the action. A unitary transformation of the electronic fields  $\psi_\sigma \rightarrow e^{-i\theta/2}\psi_\sigma$  is then accompanied by a transformation  $\psi_i \rightarrow e^{+i\theta/2}\psi_i$ , which effectively turns  $ien_iV$  into  $ien_i\Phi$ .

After the unitary transformation with  $\theta = \varphi$ , the perturbation expansion can also be done on the level of  $\mathcal{G}^{-1}$ , in terms of the real field  $\Delta_1$  and

$$\tilde{K} = \frac{m}{2}\mathbf{v}_s^2 - ie\Phi; \quad \tilde{L} = \frac{i}{2}\{\nabla, \mathbf{v}_s\}. \quad (33)$$

The whole derivation of the effective action is completely analogous and even slightly easier. In this way, one again recovers the result (28), now without making use of the assumption about small electromagnetic potentials and phase. These considerations emphasize the remarkable role of gauge invariance and conclude our derivation of the effective action for a BCS superconductor.

### 3 The normal metal and Ginzburg-Landau theory

#### 3.1 The normal metal

As a first application of equation (28), we will consider the normal metal limit for temperatures above the critical temperature  $T_C$ . If one puts  $\Delta_0 = 0$  in the normal metal and discards fluctuations of the energy gap, the electronic polarization terms can be expressed in terms of the locally gauge invariant field strengths  $\mathbf{E}$  and  $\mathbf{B}$  only, as is evident from equation (28) since in the normal metal  $S_{sc} \equiv 0$ . This contrasts to the superconducting case where also terms in  $\Phi$  and  $\mathbf{v}_s$  survive. For the normal metal we obtain

$$S_{nm} + S_{em} = \int d\mathbf{q} \left( \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} - \frac{e^2}{\mathbf{q}^2} g_0 \mathbf{E}^2 + \frac{e^2}{m^2 c^2} (g_3 - g_2) \mathbf{B}^2 \right). \quad (34)$$

This action describes standard metal physics in the RPA approximation, as is discussed for instance in reference [37]. Its analysis is most conveniently done in the Coulomb gauge. After an analytic continuation to real frequencies  $i\omega_\mu \rightarrow \omega + i\delta$  and  $|\omega_\mu| \rightarrow -i\omega$ , the zeroes of the propagators describe modes of the electronic system. Two useful limits are the clean and dirty limit, in which  $\omega, D\mathbf{q}^2 \gg \tau_r^{-1}$  and  $\omega, D\mathbf{q}^2 \ll \tau_r^{-1}$  respectively. Here  $D = v_F^2 \tau_r / 3$  is the diffusion constant with a single particle relaxation time due to impurity scattering  $\tau_r$  and Fermi velocity  $v_F$ . For frequencies of the order of the Fermi energy or the plasma frequency, metallic systems are always in the clean limit.

In the dirty limit,  $g_0$  is given by equation (B.11) and the part of the action related to the longitudinal electric field has the form

$$S_{nm} + S_{em} = \int d\mathbf{q} \left( 1 + \frac{8\pi e^2 N_0 D}{|\omega_\mu| + D\mathbf{q}^2} \right) \frac{\mathbf{E}^2}{8\pi}. \quad (35)$$

Since  $8\pi e^2 N_0 = k_{TF}^2$ , with Thomas-Fermi wavevector  $k_{TF}$ , the low frequency part  $|\omega_\mu| \ll D\mathbf{q}^2$  of this action describes metallic screening. In the opposite high frequency limit the second term of the action describes dissipation [38]: in terms of the conductivity  $\sigma = 2e^2 N_0 D$ , the second part reads  $\sigma \mathbf{E}^2 / 2|\omega_\mu|$ .

In the clean high frequency limit  $\omega \gg \tau_r^{-1}, v_F q$ , the kernel  $g_0$  is given by equation (B.7). Using  $g_0 \approx -N_0 v_F^2 \mathbf{q}^2 / (3\omega^2)$ , the longitudinal plasmon at frequency  $\omega_p^2 = 4\pi e^2 n / m$  is recovered.

The remaining part of equation (34), related to the magnetic and transverse electric fields, describes purely transverse physics. The kernel  $g_3$  can be approximated for low momenta and frequencies as  $g_3 = (p_F^2 / 3\mathbf{q}^2) g_0 + N_0 / 12 + \dots$ , and in the normal state the kernel  $g_2$  can be expressed through the Ward-identities (A.6, A.7) as  $g_2 = -nm / 2\mathbf{q}^2 + (m\omega / \mathbf{q}^2)^2 g_0$ . In the dirty limit the bubbles  $g_2$  and  $g_3$  almost cancel. The remaining action for the transverse vector potential in the dirty limit is

$$S_{nm} + S_{em} = \int d\mathbf{q} \left( q^2 + \frac{\omega_\mu^2}{c^2} + \frac{4\pi\sigma\omega_\mu^2/c^2}{|\omega_\mu| + D\mathbf{q}^2} \right) \frac{\mathbf{A}^2}{8\pi}. \quad (36)$$

The low frequency limit describes the normal skin-effect, *i.e.*,  $i\omega = c^2 \mathbf{q}^2 / (4\pi\sigma)$ , for wavelengths larger than the mean free path  $l = v_F \tau_r$ .

In the opposite clean limit and for small frequencies  $\omega_\mu \ll v_F q$ , we find explicitly that  $g_3 - g_2 \approx N_0 / 12 + \pi N_0 |\omega_\mu| / (4v_F q^3) + \dots$ . The dispersion is now different,  $i\omega = 4v_F \lambda_L^2(0) |q|^3 / 3\pi$ , with the zero temperature London length given by  $\lambda_L^{-2}(0) = 4\pi e^2 n / mc^2$ , and is related to the anomalous skin-effect. Landau diamagnetism is recovered from the small constant term in  $g_3 - g_2 = N_0 / 12 + \dots$ , and this is a  $(v_F / c)$  correction to the “ $1/8\pi$ ” in  $S_{em}$  in equation (27):  $\mu^{-1} = 1 + 1 / (4\lambda_L^2(0) p_F^2)$ . Pauli paramagnetism is not present in the action (28), since we did not include a Zeeman coupling in the original model (3). Finally, two transverse light modes with dispersion  $\omega_\mu^2 = \omega_p^2 + c^2 \mathbf{q}^2$  are present in the high frequency clean limit.

#### 3.2 The Ginzburg-Landau expansion

In the normal metal close to  $T_C$ , fluctuations of the order parameter  $\Delta$  can be studied using Ginzburg-Landau theory [39, 40], or its time dependent (TDGL) generalization [20, 41–43]. Within our present formalism, the TDGL effective action is readily derived starting from the action (28). The expansion in  $\Phi$  and  $\mathbf{v}_s$  is already performed and we need only to expand all terms in powers of  $\Delta$ . In addition, we need parts of the third and fourth order terms in the expansion of the inverse Green’s function (13). They are calculated using the normal metal Green’s functions with zero energy gap, see Appendix B3. This procedure is quite standard [40] (see also Eqs. (B.7, B.11)).

The superconducting part of the action takes the form

$$S_{\text{sc}} = N_0 \frac{1}{\beta} \sum_{\omega_\mu} \int d\mathbf{x}^3 \left( \frac{\pi |\omega_\mu|}{8T} |\Delta|^2 \right) \quad (37)$$

$$+ N_0 \int d\mathbf{x} \left( \bar{\Delta} \left[ \ln \left( \frac{T}{T_C} \right) - \xi^2(0) \nabla^2 + 4m^2 \xi^2(0) \mathbf{v}_s^2 \right. \right.$$

$$\left. \left. + \Gamma (\partial_\tau - i2eV) + 2e^2 b \Phi^2 \right] \Delta + \frac{b}{2} |\Delta|^4 \right).$$

Here the coherence length is  $\xi^2(0) = \frac{\pi}{8T} D$  in the dirty limit and  $\xi^2(0) = (7\zeta(3)/48\pi^2)(v_F/T)^2$  in the clean limit. The fourth order coefficient is  $b = 7\zeta(3)/(8\pi^2 T^2)$  [40]. The small coefficient  $\Gamma = N'_0/2\lambda N_0^2$  arises due to particle hole asymmetry [41] and is usually neglected. The term containing this coefficient describes the small difference between electronic and ionic densities resulting from the fluctuations of the order parameter. It occurs in the quoted gauge invariant form, since the time derivative from the Cooperon combines with the term  $\sim \Delta^2 V$  from the third order expansion, see equation (B.13). We will see in the next section that the same term arises also at low temperatures.

The term involving the second order spatial derivative can be combined with the  $\mathbf{v}_s^2 \Delta^2$  term from the fourth order terms in the expansion into one gauge invariant second order derivative  $\xi^2(0) |(\nabla - i\frac{2e}{c} \mathbf{A}) \Delta|^2$ . The  $\Phi^2 \Delta^2$  term, however, does not straightforwardly combine with time derivatives into gauge invariant time derivatives. The reason is that close to  $T_C$  dissipative and Hamiltonian frequency dependences mix. As an example, the dissipative  $|\omega_\mu|$  term [38] in equation (37), which turns into the dissipative time derivative of the real time TDGL equation after an analytic continuation, clearly cannot be made gauge invariant. Since the second and higher order time derivatives are usually irrelevant as compared to the dissipative  $|\omega_\mu|$  term, we do not include them in equation (37).

Let us also note that the expression (37) is correct only in the limit of low frequencies and wave vectors  $\omega_\mu, Dq^2 < 4\pi T$ . For larger frequencies the expression becomes more complicated, as we can no longer expand the kernel  $h_0$  in  $\omega_\mu/4\pi T$ , *e.g.*,  $\Psi(1/2 + \omega_\mu/4\pi T) - \Psi(1/2) \rightarrow \pi \omega_\mu/8T$  ( $\Psi$  is the digamma function, see Appendix B). Also the gradient terms in equation (37) should be modified in this case. As the corresponding expressions turn out to be quite tedious we do not present them here. For some problems, however, these modifications become significant, especially because the validity of the GL expansion (37) is restricted to temperatures  $T \sim T_C$ , in which case  $\omega_\mu$  is never really smaller than  $4\pi T$ .

The action  $S_{\text{nm}}$  is also important for the description of the dynamical properties of the superconductor and describes dissipation in the “sea” of the remaining normal electrons. It explicitly depends on the order parameter, since also the polarization bubbles can be expanded in  $\Delta$ , giving rise to additional contributions. Expanding the bubble  $g_0 + f_0$  (see Eqs. (B.9, B.10)) up to the second order in  $\Delta$ , we obtain in the limit of small frequencies and

wave vectors [15,16]

$$g_0 + f_0 = -N_0 \frac{Dq^2}{|\omega_\mu|} - N_0 \frac{\pi \Delta^2}{4|\omega_\mu|T}. \quad (38)$$

For the part of the effective action concerned with the electric field, we find

$$S_{\text{nm}} + S_{\text{em}} = \int \bar{d}q \left( 1 + \frac{4\pi\sigma}{|\omega_\mu|} + \frac{2\pi^2 e^2 N_0}{q^2 |\omega_\mu| T} \Delta^2 \right) \frac{\mathbf{E}^2}{8\pi}. \quad (39)$$

The dependence of the action  $S_{\text{nm}}$  on  $\Delta$  is important in dirty superconductors and accounts for the Maki-Thompson fluctuation enhancement of the conductivity near  $T_C$  [15,16]. The Azlamazov-Larkin fluctuation correction to the conductivity is already present in the TDGL action  $S_{\text{sc}}$  (37) due to the presence of the  $\mathbf{v}_s^2 \Delta^2$  term [14].

Extensions of the elegant TDGL action into the superconducting phase have turned out to be hard [21]. Only in the presence of a large amount of paramagnetic impurities [42] or very close to  $T_C$  [43] is this possible.

## 4 Dynamics at lower temperatures

### 4.1 Electromagnetism

We now evaluate the contents of equation (28) in the superconducting state with an equilibrium gap  $\Delta_0$ . We first focus on the parts involving  $\mathbf{E}$  and  $\Phi$  that are related to the electric field and the Josephson relation. For the prefactor of the  $\Phi^2$  term, the Josephson susceptibility, we take the bubble  $\chi_J = 8\Delta_0 k_0 / i\omega_\mu = 2N_0 n_s / n$  at zero frequency and momentum. At zero frequency and small momentum the bubble that multiplies  $\mathbf{E}^2$  is  $\chi_E = 2e^2 g_1 / (mi\omega_\mu) \approx 2e^2 N_0 n_n / (nq^2)$ . The action reads

$$S_{\text{eff}} = \int \bar{d}q \left( \left[ \frac{1}{8\pi} + \frac{e^2 N_0 n_n}{q^2} \right] \mathbf{E}^2 + e^2 N_0 \frac{n_s}{n} \Phi^2 \right). \quad (40)$$

It describes metallic screening of the electrostatic potential  $V$  with the full electronic density  $n$ , to which both terms in equation (40) contribute, and superconducting screening with superfluid density  $n_s$ , only through the second term, to enforce the Josephson relation  $V = \dot{\phi}/2e$ . At higher frequencies and momenta, weight shifts from the term  $\sim n_s$  to the term  $\sim n_n$  in such a way that the plasma frequency and Thomas-Fermi screening length remain constant. The higher order frequency and momentum dependence of the kernels is not easy to extract. For low frequency and momentum, one typically finds corrections of order  $\omega_\mu/\Delta_0$  and  $v_F q/\Delta_0$  or  $Dq^2/\Delta_0$ . In the opposite limit, for high frequencies and momenta, the kernels reduce to their normal state form, and we recover dissipation in the dirty limit for frequencies  $\Delta_0 < \omega_\mu < \tau_r^{-1}$  (*cf.* Eq. (35)). In the clean limit such an intermediate frequency regime does not exist.

We now turn to the parts of equation (28) that are related to the magnetic field and the superfluid



velocity. To lowest order in the external momentum  $g_3 \approx g_2 \approx (\mathbf{p}_F^2/3\mathbf{q}^2)g_0 + \dots$  and  $f_3 \approx f_2 \approx (\mathbf{p}_F^2/3\mathbf{q}^2)f_0 + \dots$ , so that the combination  $\chi_M \sim g_3 + f_3 - g_2 - f_2$  is equal to zero in the  $q \rightarrow 0$  limit. What remains is the evaluation of the London susceptibility  $\chi_L = -8m\Delta_0 k_1 = mn_s$  at zero frequency and momentum, and we obtain

$$S_{\text{eff}} = \int \tilde{d}q \left( \frac{\mathbf{B}^2}{8\pi} + \frac{mn_s}{2} v_s^2 \right). \quad (41)$$

This action describes transverse screening of the magnetic field in a superconductor and is related to the London theory.

Summarizing: at high frequencies  $\omega_\mu \gg \Delta_0$  and momenta  $q \gg \xi^{-1}$  the electromagnetic properties are those of a normal metal. At low frequencies and momenta a superconductor screens, in addition to electric fields, also magnetic fields.

## 4.2 Dynamics of the energy gap and particle-hole asymmetry

The dynamics of the fluctuations of the amplitude of the energy gap  $\Delta_L$  are governed by the combination  $\chi_A = [2\lambda^{-1} + 2f_0(q) + h_0(q) + h_0(-q)]$ . The corresponding mode is heavily overdamped due to the coupling to particle-hole pairs, and starts off at frequencies  $2\Delta_0$  as discussed in references [22, 23].

We now turn to the particle-hole symmetry breaking action (26). To lowest order in the frequency and momentum, we obtain  $\chi_I(q) = h_0(q) - h_0(-q) = 2N_0\Gamma i\omega_\mu$ ,  $\chi_I^\parallel(q) = -2ie[k_0(q) + k_0(-q)] = -4ieN_0\Gamma\Delta_0$ , and  $\chi_I^\perp(q) = 2[k_1(q) - k_1(-q)] = 0$ . Together with the term  $|\Delta_1|^2 V$  from the third order expansion of the effective action, we find

$$S_{\text{sc}} + S_{\text{ph}} = \int dx \left( N_0 \frac{n_s}{n} |\Delta_1|^2 - 2ieN_0\Gamma\Phi (|\Delta_0 + \Delta_1|^2 - |\Delta_0|^2) \right), \quad (42)$$

as announced in the introduction. Apparently, the contribution to the action that couples the amplitude of the gap to the chemical potential for Cooper pairs, is independent of temperature and the mean free path. Close to  $T_C$  exactly the same term, with  $\Delta_0 \equiv 0$  appeared already in the TDGL expansion (see Eq. (37)). The coupling constant  $\Gamma$  of this term is usually small, and  $S_{\text{ph}}$  is irrelevant, except for inhomogeneous problems related to vortex motion [9].

In the core of a vortex the energy gap goes to zero, and this local variation of the gap induces a local charge density modulation, for which the action  $S_{\text{ph}}$  contains the source term [35]. Moreover, due to the singular phase field around a vortex, together with the suppression of the gap in the core, the action  $S_{\text{ph}}$  gives rise to a small additional force per length on vortices,  $-2\pi N_0\Gamma\Delta_0^2 \mathbf{v}_L \times \hat{\mathbf{z}}$ , which is proportional and perpendicular to the vortex velocity  $\mathbf{v}_L$  ( $\hat{\mathbf{z}}$  is the unit vector along a vortex in the direction of the magnetic field) [9, 30]. We do not find evidence for the much larger force  $-\pi n_s \mathbf{v}_L \times \hat{\mathbf{z}}$  found in references [36, 44].

The transverse force from equation (42) includes those parts of the Magnus and Iordanskii forces that are proportional to the vortex line velocity, as well as the spectral flow contribution [48]. In the (super-) clean limit, the Kopnin-Kravtsov force due to localized quasi-particles in the vortex core adds [8]. A further non-equilibrium correction to the transverse force on a vortex was identified in reference [47]. For a review of the several forces on a vortex-line, see, *e.g.*, reference [45].

## 4.3 The uncharged limit

The limit where the electronic charge vanishes has received some attention recently [26–29]. Although this case is realized in superfluid  $^3\text{He}$ , the different order parameter symmetry makes any s-wave considerations less useful. Furthermore, the interactions between uncharged  $^3\text{He}$  atoms is very different from the electron-electron interactions as described by our starting point (3). In particular,  $^3\text{He}$  atoms are neutral and in the  $^3\text{He}$  system no background charge is present.

For completeness, however, we also discuss the uncharged limit of our model (3). Putting  $e \rightarrow 0$  in equations (24, 26), we find to lowest order in momentum and frequency for the phase part of the action

$$S_{\text{sc}} + S_{\text{ph}} = N_0 \int \tilde{d}q \left( -i\Gamma\Delta_0^2 \dot{\varphi} + \frac{n_s}{4n} \left[ \dot{\varphi}^2 + \frac{v_F^2}{3} (\nabla\varphi)^2 \right] \right), \quad (43)$$

which gives the standard acoustic Bogoliubov-Anderson mode with velocity  $v = v_F/\sqrt{3}$ . Note that the first term in the phase action equation (43) is different from the one obtained in references [26–29]. In particular, in contrast to references [26–29] no large topological term  $in_e\dot{\varphi}/2$  is present, only a much smaller term proportional to the particle-hole asymmetry. The difference can be traced to the near perfect cancellation between ionic and electronic charge densities in our case.

For superconducting Bose-liquids a large topological term is present in the effective action, since the bosonic field itself can be taken as the order parameter at low temperatures and as a result the phase is dual to the density. However, for Fermionic superconductors one rather expects that the phase of the order parameter is dual to the amplitude of the order parameter, *i.e.*, to the energy gap. As a consequence, instead of a term  $in_e\dot{\varphi}$ , we expect a term proportional to  $i\Delta^2\dot{\varphi}$  in the effective action. This is just the content of equations (42, 43), which shows that the constant of proportionality is given by the particle-hole asymmetry parameter  $\Gamma$ .

## 5 Conclusion

We have reviewed the derivation of the effective theories for BCS superconductors and discussed the corresponding dynamics of electromagnetism and the amplitude of the energy gap. Our main result (28) is a good starting point

for investigations of quantum dynamical and statistical problems in BCS superconductivity.

We have stressed the role of gauge invariance and the corresponding Ward identities that express particle number conservation. Although a perturbation expansion can violate gauge invariance, the Ward identities allowed us to obtain explicitly gauge invariant results. In particular, we have demonstrated how the Anderson-Higgs mechanism occurs within BCS theory. In contrast, the role of Galilei invariance that was stressed in references [26–29] does not seem to play an important role in real BCS materials.

Furthermore, we included the effect of particle-hole asymmetry in our considerations. We find a small topological term proportional to the particle-hole asymmetry, that leads to an additional Hall-force on vortices (apart from the Kopnin-Kravtsov, Magnus, and Iordanskii forces [45]) as discussed in references [9, 10]. Also, we have seen that the structure of the theory is essentially the same in the clean and dirty limits. In particular, the prefactor of the topological term does not depend on the electronic mean free path. The main difference between the clean and dirty limits is the presence of an intermediate dissipative regime  $\Delta_0 < \omega_\mu, v_F q < \tau_r^{-1}$  in the dirty limit. This difference shows up for instance in the quantum dynamics of low dimensional superconductors [13].

We thank G. Schön, U. Eckern, R. Fazio, D. Geshkenbein, H. Katzgraber, D. Rainer, M. Stone, G. Volovik, K.-H. Wagenblast, and G. Zimanyi for discussions on several aspects of our results. The support by the Swiss National Foundation, the Deutsche Forschungsgemeinschaft within SFB 195, NSF Grant 95-28535, INTAS Grant 93-790-ext, and the Dutch Foundation for Fundamental Research on Matter (FOM) is gratefully acknowledged. One of us (D.S.G.) also acknowledges partial support from the International Centre for Fundamental Physics in Moscow.

## Appendix A: The Ward identity

Gauge invariance gives rise to a Ward identity, which is derived in this Appendix on the level of the Green's functions in order to obtain relations between the polarization bubbles. On the level of vertex functions this Ward identity is discussed for instance in reference [2].

Let us consider the change in the Green's function upon rotating the electronic phase by  $\varphi$ . We have on the one hand (we expand in  $\varphi$ )

$$\begin{aligned} \mathcal{G}_\phi(x, x') &= e^{i\phi(x)\sigma_3/2} \mathcal{G}(x, x') e^{-i\phi(x')\sigma_3/2} \\ &= \mathcal{G}(x, x') + \delta\mathcal{G}(x, x') \end{aligned}$$

so that

$$\begin{aligned} \delta\mathcal{G}(x, x') &= \frac{i}{2} [\phi(x)\sigma_3\mathcal{G}(x, x') - \mathcal{G}(x, x')\sigma_3\phi(x')] \quad (\text{A.1}) \\ &= \frac{i}{2} \int \tilde{d}q \tilde{d}p e^{iqx} e^{ip(x-x')} \phi_q [\sigma_3\mathcal{G}_p - \mathcal{G}_{p+q}\sigma_3], \end{aligned}$$

whereas on the other hand

$$\begin{aligned} \mathcal{G}_\phi^{-1}(x, x') &= e^{i\phi(x)\sigma_3/2} \mathcal{G}^{-1}(x, x') e^{-i\phi(x')\sigma_3/2} \\ &= \mathcal{G}^{-1}(x, x') + \delta\mathcal{G}^{-1}(x, x') \end{aligned}$$

so that

$$\begin{aligned} \delta\mathcal{G}^{-1}(y, y') &= \delta(y - y') \left( -\frac{i}{2} \dot{\phi}\sigma_3 + \frac{i}{4} \{\nabla, \nabla\phi\} \hat{1} \right. \\ &\quad \left. + i\phi\Delta\sigma_+ - i\phi\bar{\Delta}\sigma_- \right) \end{aligned}$$

and

$$\begin{aligned} \delta\mathcal{G}(x, x') &= - \int dy dy' \mathcal{G}(x, y) \delta\mathcal{G}^{-1}(y, y') \mathcal{G}(y', x') \\ &= - \int \tilde{d}q \tilde{d}p e^{iqx} e^{ip(x-x')} \phi_q \mathcal{G}_{p+q} \quad (\text{A.2}) \\ &\quad \times \left[ \frac{\omega_\mu}{2} \sigma_3 - \frac{i}{2} \frac{Q}{m} \hat{1} + i(\Delta\sigma_+ - \bar{\Delta}\sigma_-) \right] \mathcal{G}_p. \end{aligned}$$

Comparison of equations (A.1, A.2) leads to

$$\begin{aligned} \sigma_3\mathcal{G}_p - \mathcal{G}_{p+q}\sigma_3 &= \\ \mathcal{G}_{p+q} [i\omega_\mu\sigma_3 + Q/m\hat{1} - \Delta\sigma_+ + \bar{\Delta}\sigma_-] \mathcal{G}_p. \quad (\text{A.3}) \end{aligned}$$

The upper left and right components of equation (A.3) read

$$\begin{aligned} G(p) - G(p+q) &= (i\omega_\mu + Q/m)G(p)G(p+q) \\ &\quad + (-i\omega_\mu + Q/m)\bar{F}(p)F(p+q) \\ &\quad - 2\Delta\bar{F}(p)G(p+q) + 2\bar{\Delta}G(p)F(p+q), \quad (\text{A.4}) \end{aligned}$$

$$\begin{aligned} F(p) + F(p+q) &= (i\omega_\mu + Q/m)F(p)G(p+q) \\ &\quad + (-i\omega_\mu + Q/m)\bar{G}(p)F(p+q) \\ &\quad - 2\Delta\bar{G}(p)G(p+q) + 2\bar{\Delta}F(p)F(p+q). \quad (\text{A.5}) \end{aligned}$$

These identities generate the Ward identities for the electronic polarization bubbles, by tracing them over the internal momentum and frequency  $p$  together with some function. The trace of equation (A.4) with 1 and  $Q/m$  immediately gives

$$\begin{aligned} 0 &= i\omega_\mu [g_0(q) - f_0(q)] + \mathbf{q}^2/m [g_1(q) + f_1(q)] \\ &\quad + 2\Delta_0 [k_0(q) - k_0(-q)] \quad (\text{A.6}) \end{aligned}$$

$$\begin{aligned} -n/2 &= i\omega_\mu [g_1(q) - f_1(q)] + \mathbf{q}^2/m [g_2(q) + f_2(q)] \\ &\quad + 2\Delta_0 [k_1(q) + k_1(-q)], \quad (\text{A.7}) \end{aligned}$$

and the trace of equation (A.5) with 1 yields

$$\Delta_0 \lambda^{-1} = \Delta_0 [f_0(q) - h_0(q)] + i\omega_\mu k_0(-q) - (\mathbf{q}^2/m) k_1(-q). \quad (\text{A.8})$$

The Ward identities equations (A.6, A.7, A.8) are a result of gauge-invariance (particle number conservation) and hold also after an impurity averaging procedure. They

can be simplified further using  $f_1 = 0$  which holds by symmetry.

Along completely analogous lines, the invariance with respect to rotations by  $\exp(i\mathbf{1}\varphi/2)$  leads to

$$\begin{aligned} i\omega_\mu f_0(q) &= \Delta_0[k_0(q) - k_0(-q)], \\ -(\mathbf{q}^2/m)f_2(q) &= \Delta_0[k_1(q) + k_1(-q)]. \end{aligned} \quad (\text{A.9})$$

## Appendix B: The polarization bubbles

In this Appendix we evaluate the polarization bubbles for broken particle-hole symmetry and summarize results in several limits [2, 17, 46].

### B.1 Clean limit

In our notation the unperturbed Green's function in momentum space reads explicitly

$$\begin{pmatrix} G & F \\ \bar{F} & \bar{G} \end{pmatrix} = \frac{1}{\omega_\nu^2 + \xi_{\mathbf{p}}^2 + \Delta_0^2} \begin{pmatrix} -i\omega_\nu + \xi_{\mathbf{p}} & \Delta_0 \\ \Delta_0 & -i\omega_\nu - \xi_{\mathbf{p}} \end{pmatrix}. \quad (\text{B.1})$$

The kernels are calculated by doing the sum over Matsubara frequencies first by contour integration. The notation  $E = \sqrt{\xi_{\mathbf{p}}^2 + \Delta_0^2}$  and  $E' = \sqrt{\xi_{\mathbf{p}+\mathbf{q}}^2 + \Delta_0^2}$  is used, as well as  $\int d\Omega$  to denote a normalized angular integration. For the bubble  $f_0$  we obtain

$$\begin{aligned} f_0(q) &= \int d\xi N(\xi) \int d\Omega \frac{1}{2EE'} \left( \frac{[1 - f_{E'} - f_E]}{\omega_\mu^2 + (E' + E)^2} S_f \right. \\ &\quad \left. + \frac{[f_{E'} - f_E]}{\omega_\mu^2 + (E' - E)^2} N_f \right). \end{aligned} \quad (\text{B.2})$$

Here  $f_E \equiv f(E)$  is the Fermi function, and  $S_f$  and  $N_f$  are

$$S_f = (E' + E)\Delta^2; \quad N_f = (E' - E)\Delta^2.$$

For the other bubbles we obtain similar expressions with

$$\begin{aligned} S_g &= [(E' + E)(\xi\xi' - EE') + i\omega_\mu(\xi'E - \xi E')], \\ N_g &= [(E' - E)(\xi\xi' + EE') - i\omega_\mu(\xi'E + \xi E')], \\ S_k &= [(E' + E)\xi\Delta + i\omega_\mu E\Delta], \\ N_k &= [(E' - E)\xi\Delta - i\omega_\mu E\Delta], \\ S_h &= [-(E' + E)(\xi\xi' + EE') + i\omega_\mu(\xi'E + \xi E')], \\ N_h &= [-(E' - E)(\xi\xi' - EE') - i\omega_\mu(\xi'E - \xi E')]. \end{aligned} \quad (\text{B.3})$$

The remaining integral over momenta can in general not be given in closed form. Let us therefore consider the simple limits. For zero external momenta and in the limit  $\omega_\mu \ll v_F q$ , the kernel  $f_0$  reduces to

$$f_0 = N_0 \int d\xi \frac{\Delta_0^2}{2E^2} \left( \frac{1 - 2f_E}{2E} + \frac{\partial f_E}{\partial E} \right) = \frac{N_0 n_s}{2 n}. \quad (\text{B.4})$$

For the others we obtain

$$\begin{aligned} g_0 &= -N_0 + \frac{N_0 n_s}{2 n} = \frac{N_0}{2} \frac{-n - n_n}{n} \\ h_0 &= -\frac{1}{\lambda} + \frac{N_0 n_s}{2 n} + N_0 \Gamma i\omega_\mu \\ k_0 &= N_0 \Gamma \Delta_0 + \frac{N_0 n_s}{4 n} \frac{i\omega_\mu}{\Delta_0}. \end{aligned} \quad (\text{B.5})$$

The first order expansion of the kernels in frequency and momentum at zero temperature reads

$$\begin{aligned} f_0 &= \frac{N_0}{2} - \frac{N_0}{12\Delta_0^2} \left( \omega_\mu^2 + \frac{v_F^2}{3} \mathbf{q}^2 \right) + \dots \\ g_0 &= -\frac{N_0}{2} + \frac{N_0}{12\Delta_0^2} \left( \omega_\mu^2 - \frac{v_F^2}{3} \mathbf{q}^2 \right) + \dots \\ h_0 &= -\frac{1}{\lambda} + \frac{N_0}{2} + N_0 \Gamma i\omega_\mu + \frac{N_0}{6\Delta_0^2} \left( \omega_\mu^2 + \frac{v_F^2}{3} \mathbf{q}^2 \right) + \dots \\ k_0 &= N_0 \Gamma \Delta_0 + \frac{N_0}{4} \frac{i\omega_\mu}{\Delta_0} \left[ 1 - \frac{1}{6\Delta_0^2} \omega_\mu^2 + \dots \right]. \end{aligned} \quad (\text{B.6})$$

In the normal metal  $T > T_C$  and  $\Delta_0 \equiv 0$  the expressions simplify considerably. For  $q \ll 2k_F$  we have, apart from  $f_0 = k_0 = 0$ ,

$$\begin{aligned} g_0 &= -N_0 \left( 1 - \frac{i\omega_\mu}{2v_F q} \ln \left[ \frac{i\omega_\mu + v_F q}{i\omega_\mu - v_F q} \right] \right) \\ h_0 &= N_0 \left[ \frac{\pi|\omega_\mu|}{8T} + \xi^2(0)\mathbf{q}^2 + i\Gamma\omega_\mu - \ln \left( \frac{2e^\gamma \omega_D}{\pi T} \right) \right], \end{aligned} \quad (\text{B.7})$$

where the clean limit coherence lengths is  $\xi^2(0) = 7\zeta(3)v_F^2/(48\pi^3 T^2)$ .

The bubbles are related to one another by the Ward identities, and also by approximate identities in the limit of low external momenta. As an example we discuss the relation between  $g_3$ ,  $g_2$ , and  $g_0$ .

$$\begin{aligned} g_3 &= \{(\mathbf{p} \times \mathbf{q})^2 / 2q^4\}_{GG} \\ &\approx \frac{q^a q^b}{q^4} \frac{1}{3} \delta^{ab} p_F^2 \{1\}_{GG} = \frac{p_F^2}{3q^2} g_0; \\ g_2 &= \{[\mathbf{q} \cdot (\mathbf{p} + \mathbf{q}/2)]^2 / q^4\}_{GG} \\ &\approx \frac{q^a q^b}{q^4} \frac{1}{3} \delta^{ab} p_F^2 \{1\}_{GG} = \frac{p_F^2}{3q^2} g_0. \end{aligned} \quad (\text{B.8})$$

Here we have used  $(1/2) \int_0^\pi d\theta \sin \theta \cos^2 \theta = 1/3$  and  $(1/2) \int_0^\pi d\theta \sin \theta \sin^2 \theta = 2/3$ , and taken the internal momenta at the Fermi energy. In doing so, one makes a slight error and it can be shown that the leading term in the difference  $g_3 - g_2$  is  $N_0/12$ , which is responsible for Landau diamagnetism.

### B.2 Dirty limit

We use the notation  $\omega = \omega_\nu$ ,  $\omega' = \omega_\nu + \omega_\mu$ ,  $W = \sqrt{\omega^2 + \Delta_0^2}$ , and  $W' = \sqrt{\omega'^2 + \Delta_0^2}$ . In the presence of impurities we replace all frequencies and the gap by  $\tilde{\omega} = \eta\omega$

and  $\tilde{\Delta}_0 = \eta\Delta_0$ , with  $\eta = [1 + 1/(2\tau_r W)]$ , in the single particle Green's functions. In particular  $\tilde{W} = W + 1/2\tau_r$  [46].

In the dirty limit it is more convenient to integrate over energy first. By reversing the order one only misses a constant  $-N_0$  in the expression for  $g_0$ , which is added later on. The integral over energy  $\xi_p$  and the angular integration are straightforward. In the dirty limit when  $\Delta_0\tau_r \ll 1$ , one expands in  $v_F q\tau_r = ql$  and  $\omega_\mu\tau_r$ . The full disorder averaged polarization bubble including the vertex correction due to impurity ladders is

$$f_0 = \pi N_0 \frac{1}{\beta} \sum_{\omega_\nu} \frac{\Delta_0^2}{WW'(W + W' + D\mathbf{q}^2)}, \quad (\text{B.9})$$

where  $D = \tau_r v_F^2/3$  is the diffusion constant. The other kernels are

$$\begin{aligned} g_0 &= -N_0 + \pi N_0 \frac{1}{\beta} \sum_{\omega_\nu} \frac{WW' - \omega\omega'}{WW'(W + W' + D\mathbf{q}^2)}, \\ h_0 &= \pi \frac{1}{\beta} \sum_{\omega_\nu} \frac{-N_0(WW' + \omega\omega') + N'_0 WW' i\tilde{\omega}_\mu}{WW'(W + W' + D\mathbf{q}^2)}, \\ k_0 &= \pi \frac{1}{\beta} \sum_{\omega_\nu} \frac{N_0(-i\omega\Delta_0) + N'_0 WW' \tilde{\Delta}}{WW'(W + W' + D\mathbf{q}^2)}. \end{aligned} \quad (\text{B.10})$$

A considerable simplification occurs for  $T > T_C$  when  $\Delta_0 \equiv 0$ . In this case the remaining sums over the internal frequencies can be carried out and yield differences of digamma functions. The low momentum and frequency expansion gives, apart from  $f_0 = k_0 = 0$ ,

$$\begin{aligned} g_0 &= -N_0 \frac{D\mathbf{q}^2}{|\omega_\mu| + D\mathbf{q}^2}, \\ h_0 &= N_0 \left[ \frac{\pi |\omega_\mu| + D\mathbf{q}^2}{8T} + i\Gamma\omega_\mu - \ln \left( \frac{2e^\gamma \omega_D}{\pi T} \right) \right]. \end{aligned} \quad (\text{B.11})$$

In this limit, the kernels  $g_0$  and  $h_0$  are nothing but the Diffuson and Cooperon. For temperatures  $T < T_C$  no simple expressions are available. However, the bubbles at zero external momentum and frequency are known

$$f_0(0) = \frac{\pi}{2} N_0 \frac{1}{\beta} \sum_{\omega_\nu} \frac{\Delta_0^2}{(\omega_\nu^2 + \Delta_0^2)^{3/2}} = \frac{N_0}{2} \frac{n_s}{n} \quad (\text{B.12})$$

See equation (B.5) for the other kernels.

### B.3 Higher order bubbles

For the construction of the Ginzburg-Landau functional, the diagrams with 3 Green's functions at zero external momenta are needed. The third order contribution  $I_3^-$  is

independent of the mean free path

$$\begin{aligned} I_3^- &= \int \tilde{d}p [G(p) - \bar{G}(p)] G(p) \bar{G}(p) \\ &= \frac{1}{\beta} \sum_{\omega_\nu} \int d\xi (N_0 + \xi N'_0 + \dots) \frac{-2\xi}{(\omega_\nu^2 + \xi^2)^2} \\ &= -2N'_0 \frac{1}{\beta} \sum_{\omega_\nu} \frac{1}{|\omega_\nu|} \int_{-\infty}^{\infty} \frac{dx x^2}{(1+x^2)^2} \\ &= -N'_0 \ln \left( \frac{2e^\gamma \omega_D}{\pi T} \right) \approx -N_0 \Gamma, \end{aligned} \quad (\text{B.13})$$

and is proportional to the particle-hole asymmetry of the problem, whereas the combination  $I_3^+$  is

$$I_3^+ = \int \tilde{d}p [G(p) + \bar{G}(p)] G(p) \bar{G}(p) = 0. \quad (\text{B.14})$$

Finally, below  $T_C$  we need the combination

$$\begin{aligned} I_3^{\text{sc}} &= \int \tilde{d}p [G(p) - \bar{G}(p)] [3F(p)F(p) + G(p)\bar{G}] \\ &\approx -2N_0 \Gamma. \end{aligned} \quad (\text{B.15})$$

## References

1. J. Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
2. J.R. Schrieffer, *Theory of Superconductivity* (Benjamin Cummings, 1964).
3. G. Schön, in *Nonequilibrium Superconductivity*, edited by D.N. Langenberg, A.I. Larkin (Elsevier, 1986).
4. C.J. Pethick, H. Smith, Annals Phys. **119**, 133 (1979).
5. R.V. Carlson, A.M. Goldman, Phys. Rev. Lett. **34**, 11 (1975); A. Schmid, G. Schön, Phys. Rev. Lett. **34**, 941 (1975); S.N. Artemenko, A.F. Volkov, Zh. Eksp. Teor. Fiz. **69**, 1764 (1975) [Sov. Phys. JETP **42**, 896 (1976)].
6. J.E. Mooij, G. Schön, Phys. Rev. Lett. **55**, 114 (1985); O. Buisson, P. Xavier, J. Richard, Phys. Rev. Lett. **73**, 3153 (1994).
7. J. Bardeen, M.J. Stephen, Phys. Rev. A **140**, 1197 (1965); P. Nozieres, W.F. Vinen, Philos. Mag. **14**, 667 (1966); L.P. Gorkov, N.B. Kopnin, Usp. Fiz. Nauk. **116**, 413 (1975) [Sov. Phys. Uspekhi **18**, 496 (1996)].
8. N.B. Kopnin, V.E. Kravtsov, Pis'ma Zh. Eksp. Teor. Fiz. **23**, 631 (1976) [JETP Lett. **23**, 578 (1976)] and Zh. Eksp. Teor. Fiz. **71**, 1644 (1976) [Sov. Phys. JETP **44**, 861 (1976)].
9. A. van Otterlo, M.V. Feigelman, V.B. Geshkenbein, G. Blatter, Phys. Rev. Lett. **75**, 3736 (1995); M.V. Feigelman, V.B. Geshkenbein, A.I. Larkin, V.M. Vinokur, JETP Lett. **62**, 834 (1995).
10. G.E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 144 (1986) [JETP Lett. **44**, 185 (1986)].
11. G. Blatter, V.B. Geshkenbein, V.M. Vinokur, Phys. Rev. Lett. **66**, 3297 (1991); B.I. Ivlev, Yu.N. Ovchinnikov, R.S. Thompson, Phys. Rev. B **44**, 7023 (1991).
12. J.S. Langer, V. Ambegaokar, Phys. Rev. **164**, 498 (1967); D.E. McCumber, B.I. Halperin, Phys. Rev. B **1**, 1054 (1970).

13. A.D. Zaikin, D.S. Golubev, A. van Otterlo, G.T. Zimanyi, Phys. Rev. Lett. **78**, 1552 (1997); Sov. Phys. Uspekhi **168**, 244 (1998).
14. L.G. Azlamazov, A.I. Larkin, Fiz. Tverd. Tela **10**, 1140 (1968) [Sov. Phys. Solid State **10**, 875 (1968)].
15. K. Maki, Phys. Rev. **141**, 331 (1966).
16. R.S. Thompson, Phys. Rev. B **1**, 327 (1970).
17. U. Eckern, F. Pelzer, J. Low Temp. Phys. **73**, 433 (1988).
18. A.M. Finkel'stein, Physica B **197**, 636 (1994); R.A. Smith, M.Yu. Reizer, J.W. Wilkins, Phys. Rev. B **51**, 6470 (1995).
19. G. Blatter, B.I. Ivlev, Phys. Rev. Lett. **70**, 2621 (1993); Phys. Rev. B **50**, 10272 (1994).
20. A. Schmid, Phys. Kondens. Materie **5**, 302 (1966).
21. E. Abrahams, T. Tsuneto, Phys. Rev. **152**, 416 (1966).
22. V.N. Popov, *Functional Integrals and Collective Excitations* (Cambridge University Press, 1987).
23. H. Kleinert, Fortschr. Phys. **26**, 565 (1978).
24. V. Ambegaokar, in *Percolation, Localization, and Superconductivity*, edited by A.M. Goldman, S.A. Wolf, *NATO-ASI Series B 109* (Plenum Press, New York, 1984).
25. H.T.C. Stoof, Phys. Rev. B **47**, 7979 (1993).
26. M. Greiter, F. Wilczek, E. Witten, Mod. Phys. Lett. B **3**, 903 (1989).
27. A.M.J. Schakel, Mod. Phys. Lett. B **4**, 927 (1990).
28. I.J.R. Aitchison, P. Ao, D.J. Thouless, X.-M. Zhu, Phys. Rev. B **51**, 6531 (1995).
29. M. Stone, Int. J. Mod. Phys. B **9**, 1359 (1995).
30. A.T. Dorsey, Phys. Rev. B **46**, 8376 (1992); N.B. Kopnin, B.I. Ivlev, V.A. Kalatsky, J. Low Temp. Phys. **90**, 1 (1993).
31. P.W. Anderson, Phys. Rev. **112**, 1900 (1958); Phys. Rev. **130**, 439 (1963); P.W. Higgs, Phys. Lett. **12**, 132 (1964).
32. See the volume *Nonequilibrium Superconductivity*, edited by D.N. Langenberg, A.I. Larkin (Elsevier, 1986).
33. V. Ambegaokar, U. Eckern, G. Schön, Phys. Rev. Lett. **48**, 1745 (1982); U. Eckern, G. Schön, V. Ambegaokar, Phys. Rev. B **30**, 6419 (1984).
34. G. Schön, A.D. Zaikin, Phys. Rep. **198**, 237 (1990).
35. G. Blatter, M.V. Feigel'man, V.B. Geshkenbein, A.I. Larkin, A. van Otterlo, Phys. Rev. Lett. **77**, 566 (1996).
36. F. Gaitan, Phys. Rev. B **51**, 9061 (1995).
37. D. Pines, P. Nozières, *The Theory of Quantum Liquids* (W.A. Benjamin, 1966).
38. A.O. Caldeira, A.J. Leggett, Phys. Rev. Lett. **46**, 211 (1981); Ann. Phys. (N.Y.) **149**, 347 (1983).
39. V.L. Ginzburg, L.D. Landau, J. Exptl. Theoret. Phys. **20**, 1064 (1950).
40. L.P. Gorkov, J. Exp. Theor. Phys. **36**, 1918 (1959) [Sov. Phys. JETP **9**, 1364 (1959)].
41. H. Fukuyama, H. Ebisawa, T. Tsuzuki, Prog. Theor. Phys. **46**, 1028 (1971).
42. L.P. Gorkov, G.M. Eliashberg, Zh. Eksp. Teor. Fiz. **54**, 612 (1968) [Sov. Phys. JETP **27**, 328 (1968)].
43. G. Schön, V. Ambegaokar, Phys. Rev. B **19**, 3515 (1979).
44. P. Ao, D.J. Thouless, Phys. Rev. Lett. **70**, 2158 (1993).
45. E.B. Sonin, Phys. Rev. B **55**, 485 (1997).
46. A.A. Abrikosov, L.P. Gorkov, I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, 1975).
47. N.B. Kopnin, Phys. Rev. B **54**, 9475 (1996).
48. G.E. Volovik, JETP Lett. **65**, 676 (1997).